indeterminancy, for the states of the octet of baryons and the decuplet of baryon resonances belong uniquely in a 56 representation of $SU(6)$, and cannot be put in a 56* representation. It is rather remarkable, though, that all $SU(3)$ symmetry adds to $SU(2)$ is the fixing of the signs of G_A/G_V and μ_1/μ_0 while leaving their absolute values unaltered.

In a subsequent paper we shall discuss in detail the implications which are involved in the present approach. In particular the approximation scheme employed will be examined and additional results will be presented.

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Unstable-Particle Scattering and the Strip Approximation

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We examine, in this paper, the problem of formulating a bootstrap calculation when one of the scattering particles is unstable. Having defined the unstable-particle scattering amplitude as an S-matrix pole residue, we go on to discuss its analytic structure and point out that it may be determined from the usual Landau rules. We conclude that although the instability of the external particle complicates the structure it does not do so too severely. Therefore, we are free to postulate that, in analogy with the stable case, the unstable-particle amplitude exhibits Regge asymptotic behavior. This assumption leads us to construct a strip approximation to the amplitude which is a crossing-symmetric superposition of Regge pole terms. We point out that this approximation exhibits, in some respects, satisfactory analytic structure. In particular it takes quite well into account certain anomalous threshold effects. It satisfies a quasi-Mandelstam representation which we use to explore the analytic structure of the corresponding partial-wave amplitudes and their continuation to arbitrary angular momentum. We use certain simple discontinuity formulas to obtain dynamical equations for the partial-wave amplitudes and are consequently able to construct, formally, a complete bootstrap scheme. Finally, we mention some difficulties and unsolved problems.

I. INTRODUCTION

At the present time qualitative success has been achieved in some simple calculations involving strongly interacting particles.[1–7] More elaborate calculations have been proposed and attempted; for example, the various forms of the strip approximation to the π-π scattering amplitude.[8–11] Most of these calculations make use of elastic unitarity. It has always been intended, however, to improve on this situation by introducing some inelastic effects explicitly. In some calculations this has already been done.[12–14]

Inelastic effects due to the presence of two-body channels can be discussed by means of a finite-matrix formalism which is a simple generalization of that used for elastic calculations.[15] Most inelastic effects, however, are associated with the presence of many-particle channels. The formalism necessary for discussing these channels exactly must involve infinite matrices of a complicated kind.[16–21] It would be convenient, therefore, to have an approximate method for dealing with many-particle systems which is as analogous as possible to that for two-particle systems. The purpose of this paper is to outline such a method.

The idea, which is not new, on which the method is based, is that the dynamics of many-body systems is dominated by resonance-resonance or particle-resonance configurations. For example, the four-pion system is, for suitable ranges of the center of mass energy, dominated by the π-ω and ρ-ρ configurations of the pions. Similarly the $\pi\pi N$ system is dominated by the π-N^* and ρ-N configurations of the particles. The experimental support for this idea may be summed up by pointing to the impressive qualitative success of even very simple isobar models.[22]

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Calculations incorporating this basic idea have already been proposed and in some cases carried out.[23–25] They are essentially of two types. The first may be characterized as the *sharp resonance approximation*.[24] In this sort of calculation the unstable particles are treated as if they were stable, the imaginary parts of the resonance masses being neglected. The resulting discontinuity equations, which are deduced, with many approximations, from multiparticle unitarity, can take on a rather awkward form especially when dealing with residual three-particle effects such as the overlap of different resonance configurations.

An example of the second type of scheme is that proposed by Lovelace[26] for a nonrelativistic system and extended by Omnes and Alessandrini[27] to one with relativistic kinematics. Here essential use is made of Green's functions rather than on-energy-shell scattering amplitudes. As pointed out by Lovelace, the use of Green's functions avoids the necessity of taking explicitly into account the prominent anomalous thresholds which appear in the on-mass-shell amplitudes. However, the use of this technique sets these schemes outside the S-matrix framework used previously for two-body calculations.

All of the points raised in the preceding paragraphs are treated, at least approximately, in the scheme proposed below. Not only is it fully relativistic but it is able, while making use of on-mass-shell amplitudes only, to take into account anomalous threshold effects and residual multiparticle effects. It is enabled to do so because dynamical information about the system is expressed in terms of Regge trajectory and residue functions.

In fact, the scheme is essentially the strip approximation, in the form proposed by Chew[9] and developed by Chew and Jones,[10] extended to a situation with unstable particles. Because of this, it should, if successful, provide a natural means of including more inelastic effects explicitly into strip approximation calculations.

In setting up the scheme use will be made of the following properties of the S matrix.

(1) The existence of poles in the complex planes of the invariant variables.

(2) The fact that the residues of these poles factorize in a significant fashion.

(3) The unitarity of the S matrix in the form of discontinuity formulas.[28]

Properties (1) and (2) are needed in order to define scattering amplitudes with unstable external particles.

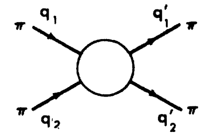

FIG. 1. π-π scattering.

The third property provides the basis for the dynamical equations which govern the system.

These points are clarified and illustrated in the following sections in terms of a model comprising three identical scalar particles (π mesons), any pair of which are subject to a resonant S-wave interaction (ρ meson). The unrealistic scalar nature of the particles is chosen to avoid having to discuss, for the time being, the additional *dynamical* complications associated with nonzero spin particles, which were pointed out by Mandelstam.[29,30] A discussion of the *physical* three-pion system cannot, of course, avoid this issue, the spin of the ρ meson being in no sense an inessential complication.

II. THE MODEL

As indicated in the introduction we shall consider, in this paper, a model theory which has as its lightest particle a scalar meson, the π meson, with mass m_π. We shall make the following assumptions:

(1) The π-π elastic-scattering amplitude exhibits an S-wave resonance, the ρ meson. The (complex) mass of the ρ meson is m_ρ.

(2) There exists a conserved multiplicative quantum number, G, called G parity where

$$G=(-1)^N$$

and N is the number of π mesons to which the state can couple. Clearly the π meson has $G=-1$ and the ρ meson has $G=+1$.

We assume, further, that the π-π amplitude has been successfully calculated in the strip approximation proposed by Chew.[9] That is, we assume that the amplitude for the process represented in Fig. 1, $A_{\pi\pi}(s,t,u)$, can be well approximated by a few Regge-pole terms:

$$\begin{aligned} A_{\pi\pi}(s,t,u)=&\sum_i[R_i^{t_1}(s,t)+R_i^{u_1}(s,u)] \\ &+\sum_i[R_i^{s_1}(t,s)+R_i^{u_1}(t,u)] \\ &+\sum_i[R_i^{s_1}(u,s)+R_i^{t_1}(u,t)], \end{aligned} \tag{1}$$

where

$$\begin{aligned} s&=(q_1+q_2)^2, \\ t&=(q_1-q_1')^2, \\ u&=(q_1-q_2')^2, \end{aligned} \tag{2}$$

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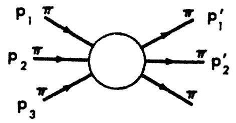

FIG. 2. Process $3\pi\to3\pi$.

$$R_i^{t_1}(s,t)=\frac{1}{\pi}\int_{t_1}^{\infty}\frac{dt'}{t'-t}R_i(t',s)\,, \tag{3}$$

$$R_i(t',s)=\tfrac{1}{2}\pi[2\alpha_i(s)+1](-q_s^2)^{\alpha_i(s)}\gamma_i(s)\times P_{\alpha_i(s)}\left(-1-\frac{t'}{2q_s^2}\right), \tag{4}$$

$$q_s^2=\tfrac{1}{4}s-m_\pi^2\,. \tag{5}$$

The Regge-trajectory functions, $\alpha_i(s)$, and the reduced pole residues, $\gamma_i(s)$, are determined self-consistently by imposing elastic unitarity on the s reaction, say, in the strip region $4m_\pi^2\leqq s\leqq s_1$. This step is performed by using the partial-wave N/D equations. The partial wave is defined, for integer l, by

$$(q_s^2)^l B_{\pi\pi}(l,s)=\frac{1}{2}\times\frac{1}{8\pi}\int_{-1}^{1}dx\,P_l(x)A_{\pi\pi}(s,x)\,, \tag{6}$$

where x is the cosine of the center-of-mass scattering angle. It is continued to complex l in the usual way by means of the Froissart-Gribov prescription. We have

$$B_{\pi\pi}(l,s)=N_{\pi\pi}(l,s)/D_{\pi\pi}(l,s)\,, \tag{7}$$

$$D_{\pi\pi}(l,s)=1-\frac{1}{\pi}\int_{4m_\pi^2}^{s_1}\frac{ds'}{s'-s}\rho_{\pi\pi}(l,s')N_{\pi\pi}(l,s')\,, \tag{8}$$

$$N_{\pi\pi}(l,s)=B_{\pi\pi}{}^P(l,s)+\frac{1}{\pi}\int_{4m_\pi^2}^{s_1}ds'\frac{B_{\pi\pi}{}^P(l,s')-B_{\pi\pi}{}^P(l,s)}{s'-s}\times\rho_{\pi\pi}(l,s')N_{\pi\pi}(l,s')\,, \tag{9}$$

$$B_{\pi\pi}{}^P(l,s)=B_{\pi\pi}(l,s)-\frac{1}{\pi}\int_{4m_\pi^2}^{s_1}\frac{\mathrm{Im}B_{\pi\pi}(l,s')ds'}{s'-s}. \tag{10}$$

The strip approximation consists in constructing $B^P(l,s)$ from the partial-wave projection of the Regge-pole approximation to $A_{\pi\pi}(s,t,u)$. By supposition one member of this self-consistent set of trajectories is associated with the ρ meson.

So much for the, formally, well understood part of the model. In the next section we go on to discuss how, in principle, the trajectory associated with the π meson may be calculated by means of the π-ρ scattering amplitude.

III. THE UNSTABLE-PARTICLE SCATTERING AMPLITUDE

Because of the assumed conservation of G parity, channels containing an even number of π mesons do not communicate with those containing an odd number. The system with lowest rest mass which couples to the π-meson, therefore, is one containing three π mesons. Relying on the belief that low mass states are more important than those of high mass, we shall suppose that the π meson is essentially a bound state of three π mesons and ignore the effects of higher thresholds. Our aim, therefore, is to calculate the bound-state poles which occur in the $3\pi\to3\pi$ scattering amplitude describing the process illustrated in Fig. 2. As is well known,[28] this amplitude has the following structure:

$$\langle p_1',p_2',p_3'|T|p_1,p_2,p_3\rangle=\frac{1}{\sqrt{3}!}\sum_{k,k'}\langle p_{i'}',p_{j'}',|T|p_i,p_j\rangle\times\delta(\mathbf{p}_k-\mathbf{p}_{k'}')2p_{k0}(2\pi)^3+A^c(p_1',p_2',p_3';p_1,p_2,p_3)\,, \tag{11}$$

where (i,j,k) and (i',j',k') are both even permutations of $(1,2,3)$.

The term A^c is referred to as the *connected* part of the amplitude. It contains no δ-function singularities. The remaining terms are called the *disconnected* parts of the amplitude.

It is convenient to regard A^c as a function of various invariant subenergies and momentum transfers. We can write

$$A^c=A^c(s;s_i;t_{ij};s_j')\,, \tag{12}$$

where

$$\begin{gathered}s_i=(p_j+p_k)^2,\quad s_i'=(p_j'+p_k')^2,\\ t_{ij}=(p_i-p_j')^2,\\ s=(p_1+p_2+p_3)^2=(p_1'+p_2'+p_3')^2,\\ (i,j,k)\text{ permutation of }(1,2,3).\end{gathered} \tag{13}$$

Of course not all of these variables are independent. They are subject to constraints; for example,

$$s_1+s_2+s_3=s_1'+s_2'+s_3'=3m_\pi^2+s\,. \tag{14}$$

They are also subject to nonlinear constraints which we shall not write out.

We shall make the standard assumption, maximal analyticity,[31] that A^c is an analytic function of its variables except where singularities are required by the unitarity equations to which the amplitude is subject. The bound state which we seek is associated with a pole of A^c at the point $s=m_\pi^2$. As has been emphasized by various authors,[32,33] these assumptions of unity and analyticity imply that the ρ-resonance pole which appears in the π-π scattering amplitude is also to be found in the complex planes of the subenergy variables (s_i,s_j').

Making use of the factorization property of the residues of these poles, we can exhibit their contribu-

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[32] D. Zwanziger, Phys. Rev. **131**, 888 (1963).
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tion to A_c; thus

$$A_c=\sum_{i,j=1}^{3}\frac{1}{\sqrt{3}}\frac{g}{s_i-m_\rho^2}\times A_{\pi\rho}(s,t_{ij})\frac{g}{s_j'-m_\rho^2}\frac{1}{\sqrt{3}}+\text{remainder}. \quad (15)$$

If we write the real and imaginary parts of m_ρ^2 explicitly,

$$m_\rho^2=s_r-i\Delta, \quad (16)$$

then unitarity implies that g, the $\rho\pi\pi$ coupling constant, satisfies approximately the equation

$$g^2=\frac{8\pi(s_r)^{1/2}}{q_r}\Delta, \quad (17)$$

where q_r is the center-of-mass momentum of the π mesons at resonance.

The factor $A_{\pi\rho}(s,t)$ is *defined* to be the π-ρ elastic scattering amplitude with t, the momentum transfer between the initial and final pions and s, the square of the center-of-mass energy. Reasons for choosing this definition of the unstable particle amplitude are:

(1) If the ρ meson were stable this definition would coincide with the usual one. The definition, therefore, puts stable and unstable amplitudes on the same footing, that is, they are both factors in pole residues.[28]

(2) The discontinuity formulas which govern the unstable amplitude so defined are analogous to those which apply to stable amplitudes. In particular, we shall be able to use an N/D method for calculating the partial waves of the π-ρ amplitude.

It is implicit in this last remark that we are going to choose the π-ρ scattering amplitude as a basis for calculating the Regge trajectory on which lies the π meson. Of course, in principle, any amplitude which contains the π-meson pole can be used in calculating it. However, we are guided by our intuitive feeling, that the π-ρ configuration dominates the 3π states, into selecting the π-ρ amplitude as a *good* starting point. We would like to emphasize that the unstable amplitude is a well-defined quantity and that we do not regard it, in this calculation, as the basis for an approximation to the complete connected 3π amplitude.

It might be thought that by concentrating on the π-ρ amplitude we cease to treat the three π mesons symmetrically. The identity of the π mesons, however, is incorporated into the formalism directly in two ways. First, if the π mesons were *not* identical there would be three distinct π-ρ channels, not just one. Correspondingly the subenergy pole residues would furnish us not with just one π-ρ amplitude but with nine which would make up a 3×3 transition matrix. The existence of only one π-ρ channel, therefore, is already a reflection of the identity of the π mesons. Secondly, we shall consider processes in which the ρ meson breaks up into two pions, namely

$$\pi+\rho\rightarrow\pi+\pi+\pi. \quad (18)$$

By treating the final-state π mesons in the correct symmetrical fashion, we shall ensure that the identical nature of these particles is acknowledged in the theory.

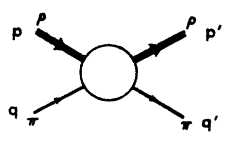

FIG. 3. π-ρ scattering.

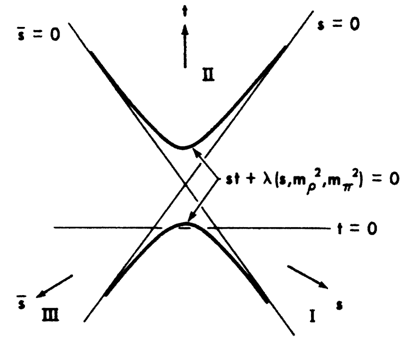

FIG. 4. Physical regions for $A_{\pi\rho}$.

The most important point, however, is that any s-plane pole, which appears in the π-ρ amplitude, must also appear in A^c. This follows simply because there are regions in the space of the subenergy variables (for example, $s_1\simeq m_\rho^2$, $s_1'\simeq m_\rho^2$) for which one of the pole terms in Eq. (15) dominates for all values of s. Therefore π-ρ bound state poles are also bound state poles of the 3π system. This, of course, is just what one expects on the intuitive physical grounds that communicating channels, stable or otherwise, share the same bound states.

To the extent, therefore, that our approximations to $A_{\pi\rho}$ are valid we can legitimately consider ourselves to be calculating, albeit approximately, bound states of the 3π system.

IV. KINEMATICS

We have defined, then, an amplitude which describes the process illustrated in Fig. 3.

$$\pi+\rho\rightarrow\pi+\rho. \quad (19)$$

This amplitude, $A_{\pi\rho}$, depends on the variables $(s,t,\bar{s})$, where

$$s=(p+q)^2=(p'+q')^2,$$
$$t=(q-q')^2=(p'-p)^2, \quad (20)$$
$$\bar{s}=(p-q')^2=(q'-p)^2,$$

and

$$s+t+\bar{s}=2(m_\pi^2+m_\rho^2). \quad (21)$$

Since m_ρ is complex we cannot straightforwardly assign a physical region to process (19). However, we shall,

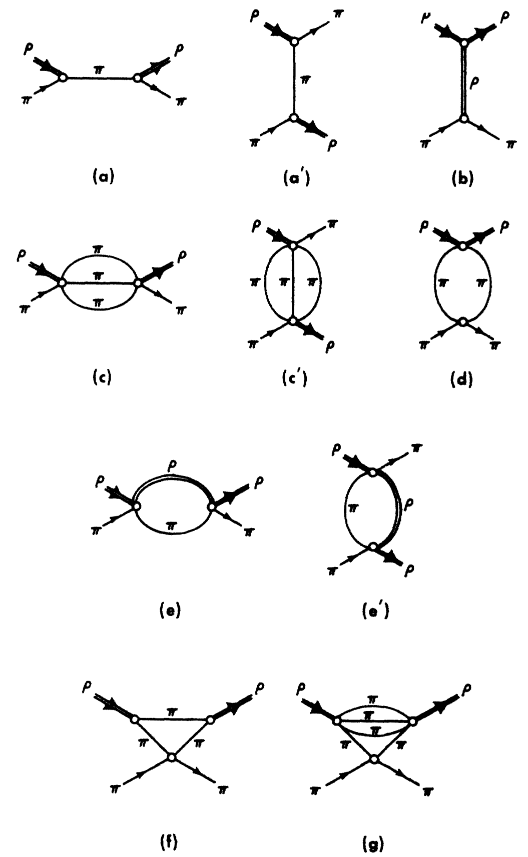

FIG. 5. Landau diagrams for singularities of $A_{\pi\rho}$.

from time to time, find it convenient to ignore the imaginary part of m_ρ in order to obtain a diagram which represents approximately the structure of $A_{\pi\rho}$. If we do so here, we find that the physical region is bounded by the straight line

$$t=0 \tag{22}$$

and the hyperbola

$$st+\lambda(s,m_\rho^2,m_\pi^2)=0 \tag{23}$$

where

$$\lambda(x,y,z,)=x^2+y^2+z^2-2(xy+yz+zx). \tag{24}$$

These curves are shown in Fig. 4. In region I of this diagram, process (19) can occur with s as the square of the center-of-mass energy and in region III with $\bar{s}$ fulfilling this role. Region II is the physical region for the process

$$\pi+\pi\rightarrow\rho+\rho \tag{25}$$

which, by crossing symmetry, is also described by $A_{\pi\rho}$ suitably continued.

V. THE ANALYTIC STRUCTURE OF $A_{\pi\rho}$

It is, by now, well understood, though not proved, that the principle of maximal analyticity implies that the singular curves of S-matrix amplitudes are given by the Landau rules.[34–39] The arguments supporting this hypothesis are essentially topological and can be applied equally well to stable- and unstable-particle scattering amplitudes.

We can infer, therefore, that $A_{\pi\rho}$ has, among others, the following singularities:

(a) $s=m_\pi^2$, (a′) $\bar{s}=m_\pi^2$,
(b) $t=m_\rho^2$,
(c) $s=9m_\pi^2$, (c′) $\bar{s}=9m_\pi^2$,
(d) $t=4m_\pi^2$,
(e) $s=(m_\rho+m_\pi)^2$, (e′) $\bar{s}=(m_\rho+m_\pi)^2$,
(f) $t=t_A=-m_\rho^2(m_\rho^2-4m_\pi^2)/m_\pi^2$,
(g) $t=t_B=-(m_\rho^2-4m_\pi^2)(m_\rho^2-16m_\pi^2)/9m_\pi^2$.

The Landau diagrams associated with these singularities are shown in Fig. 5, the lettering in the figure corresponding to that in the text. Poles (a), (a′), and (b) and normal thresholds (c), (c′), (d), (e), and (e′) are the types of singularity we expect to encounter in any amplitude of a unitary S matrix. The singularities (f) and (g) are anomalous thresholds.[35] Singularities of this type do appear in all scattering amplitudes but in the most familiar cases are hidden on an unphysical sheet. In the case of $A_{\pi\rho}$ the condition for t_A to appear on the physical sheet is[35]

$$m_\rho^2>2m_\pi^2 \tag{26}$$

and for t_B to appear is

$$m_\rho^2>10m_\pi^2. \tag{27}$$

For a ρ meson of mass equal to that of the physical particle both conditions are satisfied.

If we make use of the trick of setting $\mathrm{Im}\, m_\rho=0$, we can draw a diagram (Fig. 6) to illustrate these singulari-

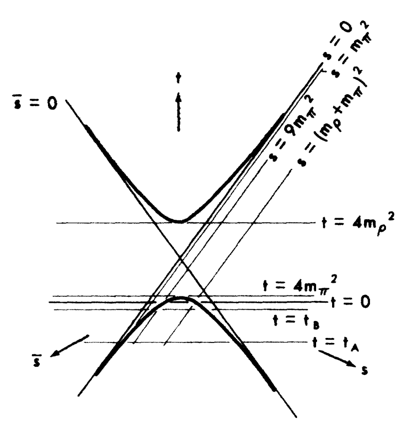

FIG. 6. Singularities of $A_{\pi\rho}$.

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[35] J. C. Polkinghorne and R. J. Eden, *Lectures in Theoretical Physics* (W. A. Benjamin and Company, Inc., New York, 1961).
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[37] J. C. Polkinghorne, Nuovo Cimento **25**, 901 (1962).
[38] H. P. Stapp, Phys. Rev. **125**, 2139 (1962).
[39] I. T. Drummond, Nuovo Cimento **29**, 720 (1963).

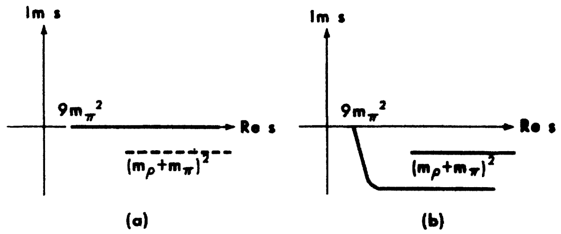

FIG. 7. Threshold structure of $A_{\pi\rho}$.

ties. We see immediately that (a), (a′), (c), (c′), (f), and (g), in contrast to the case of a stable ρ meson, all pass through one or the other of the physical regions I and III. A straightforward explanation is to be found in the fact that the instability of the ρ meson permits the corresponding Landau diagrams to represent processes in which all of the particles (even the internal ones) have physical 4-vectors. It is worth noting that the anomalous threshold, (f), is concurrent with both (a) and (a′) at points on the boundaries of the physical regions I and III. The same is true of (g) and (c) and (c′).

We can gain an idea of how these singularities can be distributed in a physical case by inserting the experimental values for m_ρ and m_π. Choosing $m_\rho^2=29m_\pi^2$ we find that

$$(m_\rho+m_\pi)^2=40.8m_\pi^2 \tag{28}$$

and

$$t_A=-725m_\pi^2. \tag{29}$$

The effect of giving m_ρ an imaginary part can be judged by using the experimental width of the ρ meson. Taking $\Delta=3.8m_\pi^2$, which corresponds to a full width at half-height of 100 MeV, we find

$$\begin{aligned}(m_\rho+m_\pi)^2&=(40.8-i4.5)m_\pi^2,\\ t_A&=(-711+i205)m_\pi^2.\end{aligned} \tag{30}$$

The change of position of the π-ρ threshold is roughly 10% while that of the anomalous threshold t_A is 30%. Ignoring the imaginary part of m_ρ, therefore, should yield a satisfactory picture of "nearby" singularities, such as the normal threshold, but can cause a considerable displacement of "far away" singularities such as the anomalous threshold. We shall find later that the discontinuity across the cut attached to t_A is considerable only while t is in the "nearby" region, so that the exact position of t_A will not be of great importance in any case. Therefore, even though we have ignored $\mathrm{Im}m_\rho$ in constructing it, the diagram in Fig. 6 gives us a good picture of the important singularities of $A_{\pi\rho}$.

In order to complete our picture of these singularities we must understand the relationship between the thresholds (c) and (e). If we draw the cut attached to (c) along the real axis, then (e) does not appear on the sheet so defined.[32,33] We can reveal the complex normal threshold by displacing the three-particle cut down into the complex s plane. This process is illustrated in Fig. 7.

We would like to draw particular attention to the situation represented in Fig. 7(b), since we make this configuration of normal threshold cuts the basis of our separation of the complete three-particle effect into a two-body effect and *residual* three-particle effects. The discontinuity across the cut attached to the complex normal threshold represents, of course, the effect of the two-body, π-ρ configuration while that across the three-particle cut, *drawn in the fashion indicated*, represents the remaining three-particle effects.

In addition to singularities which appear in only one variable, such as those discussed above, $A_{\pi\rho}$ has singular curves which depend on two variables. We discuss some of these singularities in the next section. However, we regard the singularities of this section as the most important ones. They constitute a minimum set consistent with the unitarity formulas we use.

VI. TWO-VARIABLE SINGULARITIES OF $A_{\pi\rho}$

The unstable amplitude has singular curves which are analogous to the boundaries of the Mandelstam double spectral functions of stable amplitudes. Examples of such curves are those associated with the Landau box diagrams in Fig. 8. Just as in the case of the stable amplitude, there is an infinity of types of diagrams defining singular curves of $A_{\pi\rho}$.[35] However, we shall emphasize the diagrams of Fig. 8 because they, together with the diagrams obtained by interchanging s and $\bar{s}$, represent (roughly) the physical processes taken into account in the strip approximation.

In these diagrams the lines of mass M_1, M_2, m_1, m_2, m_1', m_2', μ_1, and μ_2 represent arbitrary and, in general, multiparticle exchanges. If we make m_ρ^2 real, then provided these exchanged masses are sufficiently large so that the instability of the ρ meson becomes irrelevant, the singularity structure associated with box diagrams is in conformity with the Mandelstam representation. It follows that a large contribution to the asymptotic discontinuities in the variables $(s,t,\bar{s})$ comes from terms

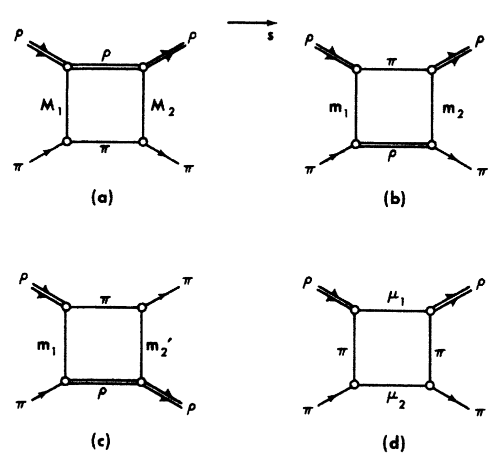

FIG. 8. Landau box diagrams.

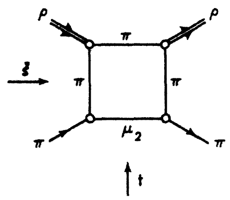

FIG. 9. Box diagrams with low internal mass.

which satisfy the Mandelstam representation. When we restore the imaginary part ot m_ρ we cannot describe the analytic structure in such a simple way. However we shall assume that the description remains substantially correct on the grounds that $\mathrm{Im}m_\rho$ is small compared with the important internal masses in the diagrams.

In order to show what happens when one of the exchanged masses is small we consider the diagram of Fig. 9. The equation for the associated curve is[35]

$$(t-4m_\pi^2)\lambda(s,m_\pi^2,\mu_2)=4\mu_2^2[(m_\rho^2-2m_\pi^2)^2+\mu_2^2m_\pi^2 \\ -(m_\rho^2-2m_\pi^2)(s-\mu_2^2-m_\pi^2)]. \quad (31)$$

The curve is drawn in Fig. 10 ($\mathrm{Im}m_\rho=0$) for two ranges of values of μ_2, namely

$$\mu_2>(m_\rho^2-2m_\pi^2)/m_\pi \quad (32)$$

and

$$\mu_2<(m_\rho^2-2m_\pi^2)/m_\pi. \quad (33)$$

In the former case the analytic structure is still consistent with the Mandelstam representation provided we include the anomalous threshold cut,

$$t_A\leqq t\leqq 4m_\pi^2, \quad (34)$$

in defining the physical sheet of the scattering amplitude. For example, the complex surfaces attached to the real curves Γ_2 and Γ_3 in Fig. 10(a) are not singular in this sheet. Since it is large values of μ_2 which are associated with the asymptotic properties of $A_{\pi\rho}$ we see that, at least in the asymptotic region, large $(s,\bar{s})$, the contributions with an anomalous threshold still have reasonably clean cut-plane analytic structure.

When μ_2 satisfies condition (33), the singular curve is no longer consistent with the Mandelstam representation even when the anomalous threshold cut is included. For example, that part of Γ_3 to the right of the point Q, where it touches the line $t=t_A$, in Fig. 10(b) is singular in the physical sheet. Another aspect of this same situation is that Γ_1 is singular in the limits $(s\pm i\epsilon,\ t\mp i\epsilon)$, but is *not* singular in the limit $(s+i\epsilon,\ t+i\epsilon)$. That is, it is not effectively singular when it passes through the physical region. Although we are not able to accommodate the analytic structure of this diagram in our approximation scheme, in a simple way we may draw some comfort from this last remark since we can use it to argue that such a contribution will not vary rapidly in the region of interest, and, consequently, will not represent a strong dynamical effect.

A simple discussion of the analytic properties of the diagram of Fig. 9 becomes impossible when we restore the imaginary part to m_ρ. We shall assume, nevertheless, that the above discussion is still a useful guide when $\mathrm{Im}m_\rho$ is small. If it turns out that it is misleading in an important way, then the following sections will have to be appropriately modified.

We can sum up our analysis of the singular curves of $A_{\pi\rho}$ by saying that, in the asymptotic region particularly, the sheet structure of the important contributions does not depart too seriously from a direct product of cut planes provided we include the anomalous threshold cuts.

VII. ASYMPTOTIC BEHAVIOR OF $A_{\pi\rho}$

Experience with stable-particle calculations has revealed the importance of understanding the asymptotic behavior of scattering amplitudes. An assumption which has gained currency is that of Regge asymptotic behavior.[40–42] It relates the high-energy dependence of an amplitude in the direct channel to families of bound states and resonances (Regge trajectories) in the crossed channels, thus permitting a systematic discussion of the subtractions necessary for dispersion relations. In addition to being theoretically attractive, the hypothesis has also provided the basis for a satisfactory fit to the experimentally observed diffraction scattering.[43,44]

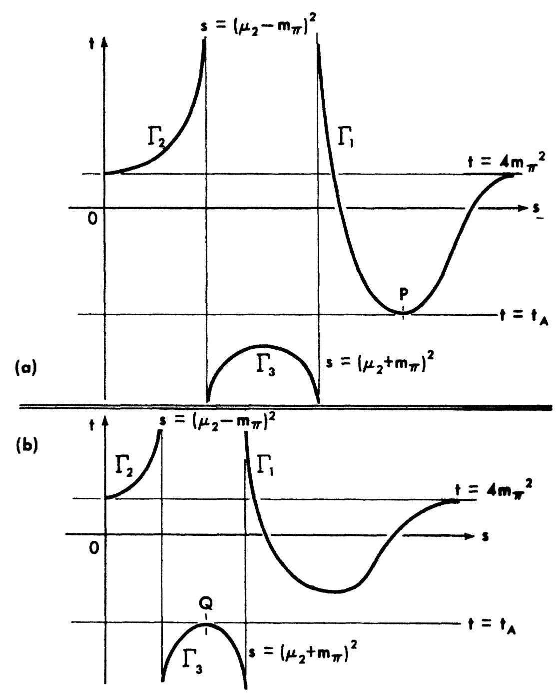

FIG. 10. Singular curves associated box diagram of Fig. 9.

[40] T. Regge, Nuovo Cimento 14, 951 (1959).
[41] T. Regge, Nuovo Cimento 18, 947 (1960).
[42] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. 126, 1202 (1962).
[43] W. Rarita and V. L. Teplitz, Phys. Rev. Letters 12, 206 (1964).
[44] R. K. Logan, Phys. Rev. Letters 14, 414 (1965).

Even if, as is presumably the case, the Regge hypothesis does not provide a complete description of asymptotic behavior it does seem appropriate to strip approximation calculations.[45] Indeed, as indicated in Sec. II, Chew has shown that it can be explicitly incorporated into such a calculation.[9,10]

Since we believe that unstable amplitudes should be discussed in essentially the same way as stable ones we assume that the asymptotic dependence of $A_{\pi\rho}$ is controlled by terms which exhibit Regge behavior and, therefore, behave like

$$\begin{aligned}&\gamma_{\pi\pi}(t)(s^{\alpha(t)}+\bar{s}^{\alpha(t)}) \quad \text{at fixed } t\,,\\ &\gamma_{\pi\rho}(s)(t^{\alpha(s)}\pm\bar{s}^{\alpha(s)}) \quad \text{at fixed } s\,,\\ &\gamma_{\pi\rho}(\bar{s})(t^{\alpha(\bar{s})}\pm s^{\alpha(\bar{s})}) \quad \text{at fixed } \bar{s}\,.\end{aligned}$$

Implicit in this assumption are two others:

(1) There exist Regge trajectories, $\alpha(s)$, which correlate the bound states and resonances of the π-ρ and, therefore, of the 3π system.

(2) Asymptotically $A_{\pi\rho}$ is, to a good approximation, analytic in the complex plane of the asymptotic variable cut along the positive real axis.

Assumption (1) that multibody trajectories exist is difficult to avoid in a relativistic theory since *all* bound states are coupled to multiparticle states. It is particularly necessary, for example, to render meaningful the identification of Regge recurrences since they lie in strongly inelastic regions. Unfortunately, proofs of the existence of such trajectories are not yet complete.[46,47]

Since we make the existence of Regge trajectories a central part of the theory, we are heavily committed to assumption (2). However we note that it does not conflict too seriously with what we have learned of the analytic structure of $A_{\pi\rho}$ in Sec. VI.

Following Chew and Jones,[9,10] we shall fulfill these assumptions by supposing that $A_{\pi\rho}$ can be well-approximated in the strips $|s|<s_1$, $|\bar{s}|<s_1$, $|t|<t_1$ by a crossing symmetric superposition of Regge poles. Thus,

$$\begin{aligned}A_{\pi\rho}(s,t,\bar{s})=&8\pi\sum_i \{R_{\pi\rho i}{}^{t_1}(s,t)+\xi_i\bar{R}_{\pi\rho i}{}^{s_1}(s,\bar{s})\}\\ &+8\pi\sum_i \{R_{\pi\rho i}{}^{t_1}(\bar{s},t)+\xi_i\bar{R}_{\pi\rho i}{}^{s_1}(\bar{s},s)\}\\ &+8\pi\sum_j \{R_{\rho\rho j}{}^{s_1}(t,s)+R_{\rho\rho j}{}^{s_1}(t,\bar{s})\}\,. \qquad (35)\end{aligned}$$

The Regge poles of the s, $\bar{s}$, and t reactions are contained in the first, second, and third brackets, respectively. The first part of the suffix on each term indicates the channel to which the pole can couple, while i and j indicate particular trajectories and $\xi_i=\pm1$ according as the signature of the trajectory is even or odd. In the t reaction, of course, only even signature trajectories occur. The quantitites s_1 and t_1 are the strip-width parameters, t_1 having the same value as in the π-π calculation, s_1 being real because of the assumed simplicity of the cut plane structure of $A_{\pi\rho}$. Included in the sums are those trajectories $\alpha_{\pi\rho}(s)$, $\alpha_{\pi\pi}(s)$ for which

$$\begin{aligned}\mathrm{Re}\,\alpha(s)&>-\tfrac{1}{2}\,,\\ \mathrm{Re}\,\alpha(t)&>-\tfrac{1}{2} \qquad (36)\end{aligned}$$

for some values of s and t.

The structure of the Regge-pole terms is given by the following equations (we suppress the suffices i and j):

$$\begin{aligned}&R_{\pi\rho}{}^{t_1}(s,t)=\frac{1}{\pi}\int_{t_1}^{\infty}\frac{dt'}{t'-t}R_{\pi\rho}(t',s)\,,\\ &R_{\pi\rho}(t',s)\underset{t'\to\infty}{\longrightarrow}\tfrac{1}{2}\pi[2\alpha(s)+1](-q_s{}^2)^{\alpha(s)}\gamma_{\pi\rho}(s)P_{\alpha(s)}(-z')\,,\\ &z'=1+(t'/2q_s{}^2)\,, \qquad (37)\end{aligned}$$

$$\begin{aligned}&\bar{R}_{\pi\rho}{}^{s_1}(s,\bar{s})=\frac{1}{\pi}\int_{s_1}^{\infty}\frac{d\bar{s}'}{\bar{s}'-\bar{s}}\bar{R}_{\pi\rho}(\bar{s},'s)\,,\\ &\bar{R}_{\pi\rho}(\bar{s}',s)\underset{\bar{s}'\to\infty}{\longrightarrow}\tfrac{1}{2}\pi[2\alpha(s)+1](-q_s{}^2)^{\alpha(s)}\gamma_{\pi\rho}(s)P_{\alpha(s)}(-z')\,,\\ &z'=\frac{\bar{s}'-m_\rho{}^2-m_\pi{}^2+2p_0q_0}{2q_s{}^2}\,, \qquad (38)\end{aligned}$$

$$\begin{aligned}&R_{\rho\rho}{}^{s_1}(t,s)=\frac{1}{\pi}\int_{s_1}^{\infty}\frac{ds'}{s'-s}R_{\rho\rho}(s',t)\,,\\ &R_{\rho\rho}(s',t)=\tfrac{1}{2}\pi[2\alpha(t)+1](-p_tq_t)^{\alpha(t)}\gamma_{\rho\rho}(t)P_{\alpha(t)}(-x')\,,\\ &x'=\frac{p_t{}^2+q_t{}^2+s'}{2p_tq_t}\,, \qquad (39)\end{aligned}$$

$$\begin{aligned}&p_0=(s+m_\rho{}^2-m_\pi{}^2)/2(s)^{1/2}\,,\quad q_0=(s+m_\pi{}^2-m_\rho{}^2)/2(s)^{1/2}\,,\\ &q_s{}^2=\lambda(s,m_\rho{}^2,m_\pi{}^2)/4s\,,\\ &p_t{}^2=(\tfrac{1}{4}t-m_\rho{}^2)\,,\quad q_t{}^2=(\tfrac{1}{4}t-m_\pi{}^2)\,, \qquad (40)\end{aligned}$$

where q_s is the center-of-mass momentum for the π-ρ system in the s reaction while q_t and p_t are the corresponding momenta for the π-π and ρ-ρ systems, respectively, in the t reaction. The quantities $\alpha(s)$, $\alpha(t)$, $\gamma_{\pi\rho}(s)$, $\gamma_{\rho\rho}(t)$ are, of course, the Regge-trajectory and reduced-residue functions. Since trajectory is common to many channels, we do not give it a suffix and rely on the arguments s, t, $\bar{s}$ to inform us about the reaction.

Note that in Eqs. (37) and (38) we have not specified the discontinuity function completely but have merely indicated its asymptotic form. The reason is that, unless other terms as well as the essential asymptotic part are included, the s-plane analyticity properties of the resulting Regge-pole terms are unsatisfactory. They acquire unacceptable cuts running between the points

[45] S. Mandelstam, Ann. Phys. (N. Y.) **21**, 302 (1963).
[46] R. L. Omnes, Phys. Rev. **134**, B1358 (1964).
[47] R. L. Omnes and V. Alessandrini, Phys. Rev. **136**, B1137 (1964).

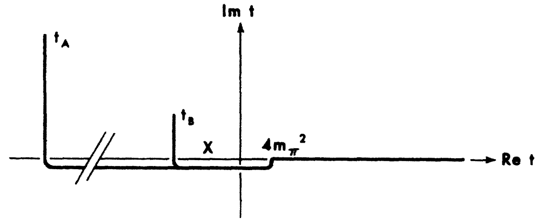

FIG. 11. Anomalous threshold structure of $A_{\pi\rho}$.

$s=0$ and $s=(m_\rho-m_\pi)^2$.[48] The origin of the difficulty is in the complicated relationship between z' and s brought about by the inequality of m_ρ and m_π.

In fact, how to construct a Regge-pole term which does not contain the unsatisfactory cut for the case of unequal masses is, at present, an unsolved problem. Since the difficulty is not connected with the instability of the ρ meson and since the precise form of the discontinuity function does not affect the discussion to follow, we shall pass over this problem even though it must be solved before a complete numerical calculation can be performed.

VIII. ANALYTIC STRUCTURE OF THE REGGE-POLE APPROXIMATION

In order to understand the analytic structure of the approximation to $A_{\pi\rho}$ provided by Eq. (35), we must know that of the trajectory and residue functions. The following properties are consistent with our dynamical equations.

Both the trajectories, $\alpha(s)$, and the residues, $\gamma_{\pi\rho}(s)$, are analytic in the s-plane cut along the real axis from $s=9m_\pi^2$ to infinity. Since it represents the angular momentum of a bound state when $s<9m_\pi^2$, we expect $\alpha(s)$ to be real analytic in the cut s plane unless two trajectories coincide at some point, a possibility we shall ignore. Since the residue, $\gamma_{\pi\rho}(s)$, is that of an amplitude obtained by an unsymmetric limiting process [Eq. (15)] we do not expect it to be real analytic. However, if $\gamma_{\pi\rho}^{(\pi)}$ is the residue of the π-meson trajectory, then

$$\gamma_{\pi\rho}^{(\pi)}(m_\pi^2)=g^2, \tag{41}$$

where g, the $\rho\pi\pi$ coupling constant, is given by Eq. (17). Since g^2 is approximately real we might guess that $\gamma_{\pi\rho}^{(\pi)}$ and also, perhaps, the other residue functions are approximately real when $s<9m_\pi^2$.

In addition to the 3π normal threshold, we expect $\alpha(s)$ and $\gamma_{\pi\rho}(s)$ to contain the complex π-ρ normal threshold and that the relationship between the two thresholds should be just that illustrated in Fig. 7.

The Regge poles which couple to the ρ-ρ channel are the same as those which couple to the π-π channel. Consequently, the trajectories, $\alpha(t)$, are real analytic in the t-plane cut from $t=4m_\pi^2$ to infinity. We shall find, however, from the dynamical equation for the t reaction that the residues, $\gamma_{\rho\rho}(t)$, while they are analytic in the same region have in addition cuts running from the anomalous thresholds $t=t_A$ and $t=t_B$ to the normal threshold at $4m_\pi^2$. This analytic structure is shown in Fig. 11.

Because of the analytic structure of $\alpha(s)$ and $\gamma_{\pi\rho}(s)$, we can verify that the Regge pole approximation to $A_{\pi\rho}$ contains the singularities (c), (c′), (e), and (e′) of Sec. V. Because of the divergence of the dispersion integrals in (37) and (38), and the corresponding integrals with s and $\bar{s}$ interchanged, the approximation also contains the poles (a) and (a′) of Sec. V. Similarly the analytic structure of the functions $\alpha(t)$ and $\gamma_{\rho\rho}(t)$ ensures that the approximation contains the singularities (d), (f), and (g) while (b) is guaranteed by the divergence of the dispersion integral in (39) together with that obtained by interchanging s and $\bar{s}$. Notice that the anomalous threshold discontinuities in this approximation are entirely the result of the corresponding discontinuities in $\gamma_{\rho\rho}(t)$. This fact will allow us to obtain explicit expressions for them.

It follows, then, that the Regge-pole approximation contains all of the singularities we identified in Sec. V as the minimum set consistent with our unitarity equations. In particular it contains the important poles of $A_{\pi\rho}$ and exhibits the corresponding Regge asymptotic behavior in the crossed channel provided, of course, the residue functions $\gamma_{\pi\rho}(s)$ and $\gamma_{\rho\rho}(t)$ vanish sufficiently fast at infinity. A sufficient condition is[49]

$$\begin{aligned}\gamma_{\pi\rho}(s)&\sim 1/s^{1/2}, \quad s\to\infty\,,\\ \gamma_{\rho\rho}(t)&\sim 1/t^{1/2}, \quad t\to\infty\,.\end{aligned} \tag{42}$$

Just as in the case of the stable amplitude, the Regge-pole approximation (35) is unsatisfactory in that it has artificial logarithmic singularities at the edge of each strip, $s=s_1$, $\bar{s}=s_1$, $t=t_1$. That is, it does not describe the transition from resonance to diffraction

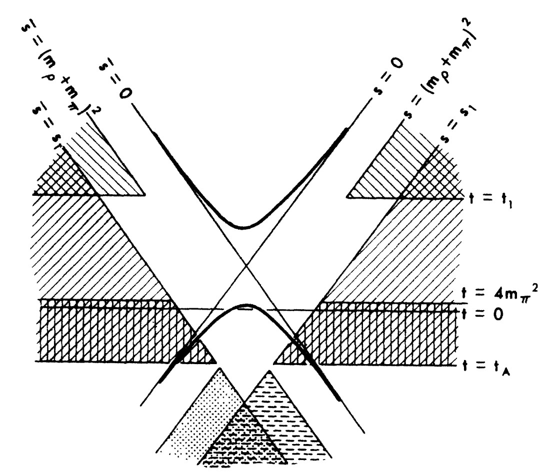

FIG. 12. Support regions for the double spectral functions.

[48] This was pointed out to the author by J. Stack.

[49] C. E. Jones, Phys. Rev. **135**, B214 (1964).

scattering in a satisfactory way. Again, just as they do for the stable amplitude, the partial-wave dynamical equations will help to mitigate this difficulty.[50]

Another defect of the approximation (35) is that it has suppressed the two-variable singularities we discussed in Sec. VI. There is no remedy for this except that of adding explicit contributions to (35) which contain whichever of these singularities we consider important.

Finally it is worth pointing out that the approximation to $A_{\pi\rho}$ we have constructed satisfies a quasi-Mandelstam representation, namely

$$A_{\pi\rho}(s,t,\bar{s})=\frac{1}{\pi^2}\int\frac{\rho_{st}(s',t')ds'dt'}{(s'-s)(t'-t)}+\frac{1}{\pi^2}\int\frac{\rho_{\bar{s}t}(\bar{s}',t')d\bar{s}'dt'}{(\bar{s}'-s)(t'-t)}+\frac{1}{\pi^2}\int\frac{\rho_{s\bar{s}}(s',\bar{s}')ds'd\bar{s}'}{(\bar{s}'-s)(s'-s)}-\frac{g^2}{s-m_\pi^2}-\frac{g^2}{\bar{s}-m_\pi^2}. \quad (43)$$

The reason for the prefix "quasi" is that some of the t' integrations run over the anomalous threshold cuts which curl up into the complex plane. However if we neglect $\mathrm{Im}m_\rho$ they lie flat and we can represent the areas in which the double spectral functions are nonzero by the diagram in Fig. 12.

IX. PARTIAL WAVES FOR π-ρ SCATTERING

The π-ρ center-of-mass scattering angle, θ, is given by the equation,

$$z=\cos\theta=1+(t/2q_s^2). \quad (44)$$

Regarding $A_{\pi\rho}$ as a function of (s,z) we can define the π-ρ partial wave amplitudes for integer angular momentum J by means of the formula

$$A(J,s)=\frac{1}{2}\times\frac{1}{8\pi}\int_{-1}^{1}dzP_J(z)A_{\pi\rho}(s,z). \quad (45)$$

We can use this equation, together with the approximate representation (43) for $A_{\pi\rho}$, to obtain a picture of the analytic structure of $A(J,s)$ in the s plane. Because the instability of the ρ meson causes crossed reaction singularities to pass through the physical region, care must be taken in specifying the integration contour in Eq. (45). We make the definition precise by requiring that the z-integration contour be the ordinary straight line

$$-1\leqq z\leqq 1 \quad (46)$$

when $s\cong(m_\rho+m_\pi)^2$, $A_{\pi\rho}$ being evaluated with t above

[50] G. F. Chew, Phys. Rev. **130**, 1264 (1963).

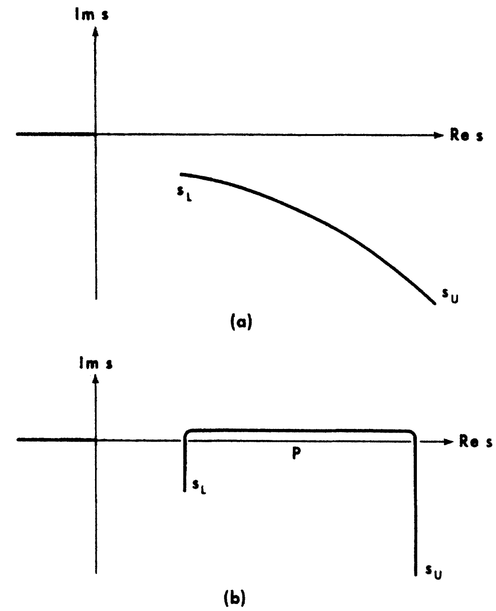

FIG. 13. Partial-wave singularities due to π exchange.

the anomalous thresholds, that is, near the spot marked X in Fig. 11. For arbitrary s we obtain $A(J,s)$ by analytically continuing the function defined near threshold. This means that z-plane singularities of $A_{\pi\rho}$ may force us to deform the integration contour.

That we do encounter such a situation can be demonstrated from an examination of the partial-wave projection of the π-exchange contribution to $A_{\pi\rho}$,

$$A_{\pi\rho}^{(\pi)}=-g^2/(\bar{s}-m_\pi^2). \quad (47)$$

Inserting this term into the right-hand side of Eq. (45) we find that its partial wave is

$$A^{(\pi)}(J,s)=-\frac{g^2}{8\pi}\frac{1}{2q_s^2}Q_J\left(\frac{m_\rho^2-2p_0q_0}{2q_s^2}\right). \quad (48)$$

This function has cuts which run between the points

$$s=0 \quad\text{and}\quad s=\infty \quad (49)$$

and between the points

$$s=s_L=2m_\rho^2+m_\pi^2 \quad\text{and}\quad s=s_U=(m_\rho^2-m_\pi^2)^2/m_\pi^2. \quad (50)$$

The latter cut, were the ρ meson stable, would be called *π-exchange short cut.* In the case of an unstable ρ meson, it lies to the right of the elastic threshold and if we use the experimental values for m_ρ it is not short. We find

$$m_\rho^2=29m_\pi^2-i3.8m_\pi^2,\quad s_L=(59-i7.6)m_\pi^2,\quad s_U=(770-i213)m_\pi^2. \quad (51)$$

If we draw this cut so that it is the image of the straight

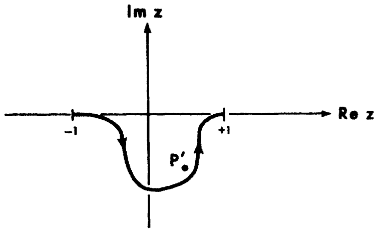

FIG. 14. Partial-wave z-integration contour. The point P' is the position of the π-exchange pole corresponding to the point P in Fig. 13.

z contour, then it lies as shown in Fig. 13(a). When we distort it into the more convenient shape of Fig. 13(b) and evaluate $A(J,s)$ at points near P in this diagram, we require a z contour of the form shown in Fig. 14.

Because of the curled anomalous threshold cuts in $A_{\pi\rho}$, the retention of a straight-line z contour would prevent us from evaluating $A(J,s)$ in a complete cut s plane. The presence of these singularities, therefore, also forces us to deform the z contour.

In order to deduce the analytic structure of $A(J,s)$ in the s plane, we use the trick first introduced by Mandelstam,[51] of continuing in the external mass variable m_ρ^2, in this case along the path K shown in Fig. 15. When m_ρ^2 has a stable value, both $A_{\pi\rho}$ and the approximation (35) satisfy the Mandelstam representation. Continuing m_ρ^2 along K ensures that the singularities move back into position in such a way that the ordinary definition of a stable partial wave passes into the definition, near threshold, which we adopted for the unstable partial wave.

It follows, then, that we can obtain the analytic structure of $A(J,s)$ by starting with that distribution of singularities appropriate to a stable ρ meson and then observing the positions to which they move when m_ρ^2 is continued along K. The result of doing this for the π-exchange contribution is already exhibited in Fig. 13(b). In Fig. 16(a) the cuts due to exchanges in the $\bar{s}$ reaction of masses greater than $9m_\pi^2$ are exhibited. Note, however, that the distribution of cuts shown, while the most convenient for clarity of exposition, is not a simple image of the real $\bar{s}$ axis. The nearly circular cut in Fig. 16(b) and the other cuts in the left-half s plane result from t-reaction exchanges of mass greater than $4m_\pi^2$. The remaining cut in the right-half s plane are reflections of the two anomalous threshold cuts.

A composite diagram showing the complete analytic structure of $A(J,s)$ would be rather unclear. Instead we show in Fig. 17 a superposition of the normal threshold cuts together with those in Fig. 13 and part of those in Fig. 16(b).

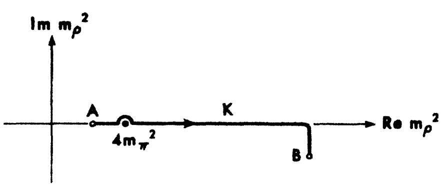

FIG. 15. Continuation path for m_ρ^2. The point A is a stable value of m_ρ^2 and B is its physical value.

[51] S. Mandelstam, Phys. Rev. Letters 4, 84 (1960).

X. CONTINUATION TO COMPLEX ANGULAR MOMENTUM

Because of the existence of the quasi-Mandelstam representation (43), we can write the approximation (35) in the form

$$A_{\pi\rho}(s,t,\bar{s})=\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dt'}{t'-t}D_t(t',s)+\frac{1}{\pi}\int_C\frac{dt'}{t'-t}D_t^A(t',s)$$
$$+\frac{1}{\pi}\int_{9m_\pi^2}^{\infty}\frac{d\bar{s}'}{\bar{s}'-\bar{s}}D_{\bar{s}}(\bar{s}',t)-\frac{g^2}{s-m_\pi^2}-\frac{g^2}{\bar{s}-m_\pi^2},\quad(52)$$

where the contour C runs along both the anomalous threshold cuts. The discontinuity functions D_t, D_t^A, $D_{\bar{s}}$ may be calculated from the functions ρ_{st}, $\rho_{s\bar{s}}$, $\rho_{\bar{s}t}$; D_t^A depending solely on the anomalous parts of ρ_{st} and $\rho_{\bar{s}t}$.

Using Eq. (52) we can define the partial waves of definite signature, $A^{(\pm)}(J,s)$, and continue them to complex J by means of a straightforward generalization of the Froissart-Gribov[52,53] method. We have then,

$$(2q_s^2)A^{(\pm)}(J,s)=\frac{1}{8\pi}\int_{4m_\pi^2}^{\infty}dt'Q_J(z')D_t(t',s)$$
$$+\frac{1}{8\pi}\int_C dt'Q_J(z')D_t^A(t',s)$$
$$\pm\frac{1}{8\pi}\int_{qm_\pi^2}d\bar{s}'Q_J(z'')D_{\bar{s}}(\bar{s}',s)$$
$$\pm\frac{g^2}{8\pi}Q_J\left(\frac{2p_0q_0-m_\rho^2}{2q_s^2}\right),\quad(53)$$

where

$$z'=1+t'/2q_s^2,$$
$$z''=(\bar{s}'-m_\rho^2-m_\pi^2+2p_0q_0)/2q_s^2.\quad(54)$$

The direct channel pole is reproduced by a suitable divergence of the infinite integrals. Of course, care has to be taken in the evaluation of these integrals since the instability of the ρ meson results in an overlap of the cuts of the Q_J functions and the integration contours. However, the difficulties are ones of practice rather than principle. Possible ambiguities in the expressions in Eq. (53) may be resolved by resorting to the trick of continuing m_ρ^2 along the path K in Fig. 15 and following the evolution of the integration contours.

[52] M. Froissart, Report to the LaJolla Conference on Theoretical Physics, June 1961 (unpublished).
[53] V. N. Gribov, Zh. Eksperim. i Teor. Fiz. 41, 667 and 1962 (1961) [English transl.: Soviet Phys.—JETP 14, 478 and 1395 (1962)].

As a result of using this prescription the integrations over the contour C, for example, contain contributions for which $Q_J(z')$ must be evaluated on a sheet obtained by continuing clockwise through the cut

$$-\infty < z' < -1.$$

It follows that $A^{(\pm)}(J,s)$ does not vanish as J becomes infinite and hence that a Sommerfeld-Watson transform for $A_{\pi\rho}$ does not exist. The virtue of this method, however, for continuing to complex J is that it preserves the essential relationship between asymptotic behavior and poles in the J plane.

Finally we define, in the standard manner, reduced amplitudes $B^{(\pm)}(J,s)$ by

$$B^{(\pm)}(J,s) = A^{(\pm)}(J,s)/(q_s^2)^J. \tag{55}$$

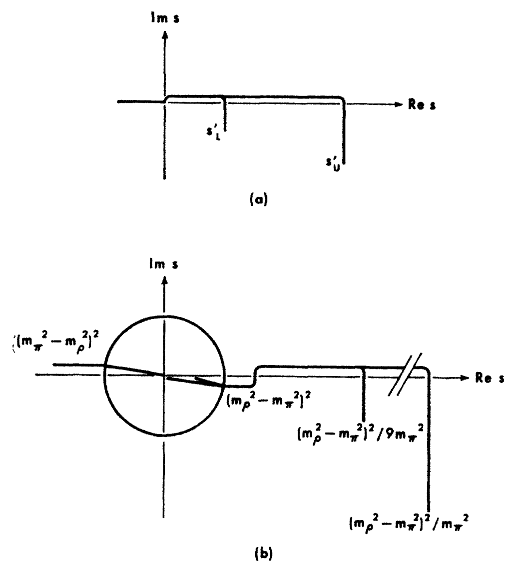

FIG. 16. (a) Cuts due to $\bar{s}$ exchanges with $\bar{s} \geq 9m_\pi^2$. We have $s_L' = 2m_\rho^2 - 7m_\rho^2$ and $s_U' = (m_\rho^2 - m_\pi^2)^2/9m_\pi^2$. (b) Cuts due to t exchanges.

They have the same s-plane analytic structure as that which we deduced for $A(J,s)$, J an integer.

XI. PARTIAL WAVES FOR $\pi+\pi \to \rho+\rho$

The center-of-mass scattering angle ϕ for the process

$$\pi+\pi \to \rho+\rho$$

is given by

$$x = \cos\phi = (p_t^2 + q_t^2 + s)/2p_t q_t. \tag{56}$$

Putting $a(t,x) = A_{\pi\rho}(s,t,\bar{s})$ the partial waves for this process are defined, for integer angular momentum l, by

$$a(l,t) = \frac{1}{2} \times \frac{1}{8\pi} \int_{-1}^{1} dx\, P_l(x) a(t,x). \tag{57}$$

The representation (43) allows us to deduce the t-plane

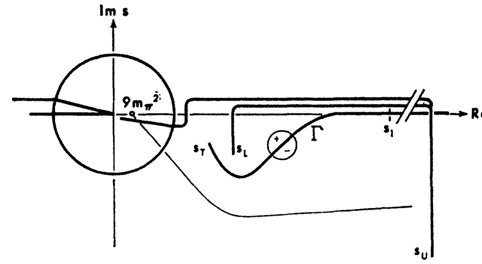

FIG. 17. Relationship of the normal threshold cuts to some of the π-ρ partial wave s-plane singularities due to crossed reaction singularities.

analyticity properties of $a(l,t)$. They are shown in Fig. 18.

We can also deduce that $A_{\pi\rho}$ can be put in the form

$$A_{\pi\rho}(s,t,\bar{s}) = -\frac{g^2}{s-m_\pi^2} + \frac{1}{\pi}\int_{9m_\pi^2}^{\infty} \frac{ds'}{s'-s} D_s(s',t) - \frac{g^2}{\bar{s}-m_\pi^2} + \frac{1}{\pi}\int_{9m_\pi^2}^{\infty} \frac{d\bar{s}'}{\bar{s}'-\bar{s}} D_s(\bar{s}',t), \tag{58}$$

where D_s can be deduced from ρ_{st} and $\rho_{s\bar{s}}$. Again, we use the Froissart-Gribov method for continuing to complex l. We have

$$(p_t q_t) a(l,s) = \frac{g^2}{8\pi} Q_l\left(\frac{p_t^2 + q_t^2 + m_\pi^2}{2p_t q_t}\right) + \frac{1}{8\pi}\int_{9m_\pi^2}^{\infty} ds' Q_l(x') D_s(s',t), \tag{59}$$

where

$$x' = \frac{p_t^2 + q_t^2 + m_\pi^2}{2p_t q_t}. \tag{60}$$

Since only positive signature amplitudes exist for the t reaction, we have not used the $(\pm)$ superfix. In contrast to the situation we described in the previous section, the evaluation of the Q_l functions presents no problem when $t > 4m_\pi^2$. Again the important connection between asymptotic behavior and poles in the l plane is preserved by this continuation.

We define the reduced partial-wave amplitude by

$$b(l,t) = a(l,t)/(p_t q_t)^l. \tag{61}$$

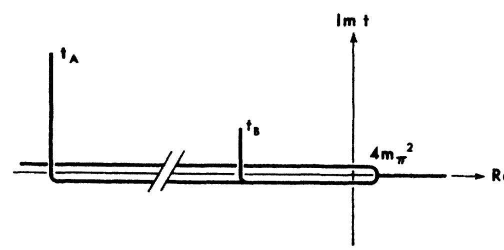

FIG. 18. Singularities in the t plane of partial waves for the process $\pi+\pi \to \rho+\rho$.

The analytic structure of $b(l,t)$ is, of course, just that represented in Fig. 18.

XII. DYNAMICAL EQUATIONS FOR $B^{(\pm)}(J,s)$

The dynamical equations governing π-ρ scattering are based on the formula connecting the discontinuity of $B(J,s)$ [we drop the $(\pm)$ superfix] across the π-ρ normal threshold cut, to the amplitude itself. We have,

$$B(J,s_+)-B(J,s_-)=2i\rho(J,s)B(J,s_+)B(J,s_-)\,, \quad (62)$$

where the significance of $+$ and $-$ is indicated in Fig. 17 and

$$\rho(J,s)=q_s^{2J+1}/s^{1/2}. \quad (63)$$

This formula is completely analogous to the one applicable to stable particle scattering amplitudes. Although we have not done so, we believe that we could derive it from the unitarity equations governing the $3\pi\to 3\pi$ amplitude. Certainly it is consistent with the results of investigations of multiparticle unitarity.[32] An heuristic proof can be found by supposing that the ρ meson is stable, writing down Eq. (62) and then continuing both sides in m_ρ^2 along the path in Fig. 15. Only if another singularity of $B(J,s)$ were to pass through the threshold $s=(m_\rho+m_\pi)^2$ would the proof break down. However, what we have learned of the other singularities of $B(J,s)$ suggests that this phenomenon does not occur.

We can define a (complex) phase shift $\delta(J,s)$ by the equation

$$\exp\{2i\delta(J,s)\}=1+2i\rho(J,s)B(J,s)\,. \quad (64)$$

Any ambiguities are resolved by requiring that

$$\delta(J,(m_\rho+m_\pi)^2)=0 \quad (65)$$

provided that there are no poles in the sheet defined by the configuration of cuts in Fig. 17. Otherwise the function is defined by analytic continuation.

In situations where the threshold condition (65) holds we can, following Omnes,[54] define a D function by

$$D(J,s)=\exp\left\{-\frac{1}{\pi}\int_{(m_\rho+m_\pi)^2}^{s_1}\frac{ds'}{s'-s}\delta(J,s')\right\} \quad (66)$$

the integral being evaluated with the contour Γ of Fig. 17.

It is easy to verify from Eq. (62) that if

$$N(J,s)=B(J,s)D(J,s) \quad (67)$$

then $N(J,s)$ does not have any discontinuity along the contour Γ between threshold and the edge of the strip $s=s_1$.

By following procedures already established for stable amplitudes, we can show that

$$D(J,s)=1-\frac{1}{\pi}\int_{(m_\rho+m_\pi)^2}^{s_1}\frac{ds'}{s'-s}\rho(J,s')N(J,s') \quad (68)$$

[54] R. L. Omnes, Nuovo Cimento **8**, 316 (1958); **21**, 524 (1961).

and

$$N(J,s)=B^P(J,s)+\frac{1}{\pi}\int_{(m_\rho+m_\pi)^2}^{s_1}ds'\times\frac{B^P(J,s')-B^P(J,s)}{s'-s}\rho(J,s')N(J,s')\,, \quad (69)$$

where

$$B^P(J,s)=B(J,s)-\frac{1}{2\pi i}\int_{(m_\rho+m_\pi)^2}^{s_1}\frac{ds'}{s'-s}[B(J,s')]_s \quad (70)$$

and $[B(J,s)]_s$ is the discontinuity of $B(J,s)$ across the normal threshold cut, Γ. Of course, this contour, Γ, also determines the integration paths in Eqs. (68), (69), and (70).

Just as in the case of stable-particle scattering we obtain a closed set of dynamical equations by supposing that a good estimate of $B^P(J,s)$ can be obtained by subtracting out the normal threshold contribution to the partial waves of the approximation (35).[9,10] Again, just as in the stable case, our equations guarantee that the discontinuity of $B(J,s)$ is continuous across the strip boundary $s=s_1$.

The Regge trajectories are obtained as the zeros of D, that is

$$D(\alpha(s),s)=0 \quad (71)$$

and the residue functions are obtained from the equation

$$\gamma_{\pi\rho}(s)=[N(J,s)/(\partial/\partial J)D(J,s)]_{J=\alpha(s)}\,. \quad (72)$$

The dynamical calculation $B(J,s)$ can, then, be carried out in a manner analogous to that for a stable amplitude. Differences from the stable case are:

(i) A complex contour is used in formulating the integral equation for N.

(ii) The distribution of cuts and singularities in B^P is rather complicated. Some of them lie near the π-ρ normal threshold cut thus requiring that great care be taken in evaluating integrals along Γ.

(iii) One of the cuts of B^P, namely that attached to the 3π threshold, is a "unitarity" cut rather than a "force" cut. Experimentally, as was emphasized in the introduction, the effect of this threshold on cross sections is not discernible. We might hope, therefore, that this particular discontinuity is not *dynamically* important either.

An exceptional situation arises, however, when a resonance pole moves to the left of the π-ρ threshold. Its presence automatically implies that the 3π discontinuity in the neighborhood of the pole is considerable. Of course, we are free to sweep the cut away from the pole, *except when it passes through the 3π threshold.* The way in which the cut influences the motion of the pole in the neighborhood of this singularity is discussed in the next section.

XIII. THE ANALYTICITY OF N AND D IN THE J PLANE

Jones[55] has pointed out that the exact N and D functions must each have a cut in the J plane between the points $J=\alpha(s_{1+})$ and $J=\alpha(s_{1-})$, the discontinuity across the cut vanishing when the ratio N/D is formed. Analogous remarks are, of course, applicable to the unstable amplitude.

The N and D functions for the π-ρ amplitude exhibit another cut of a similar nature. It occurs to compensate for the absence from the D function of a branch point at the 3π threshold. If $D(J,s)$ were analytic in the J plane, the trajectory functions predicted by equation (71) would contain only a π-ρ normal threshold and could never, therefore, be real analytic below $s=9m_\pi^2$. We can deduce, then, that $D(J,s)$ has a J-plane branch at

$$J=\alpha(9m_\pi^2). \tag{73}$$

The branch point originates in $B^P(J,s)$ and propagates itself through the N/D equations to $N(J,s)$ and $D(J,s)$. We can see this explicitly by noting that, in the presence of a pole on the sheet defined in Fig. 17, the definition of $B^P(J,s)$, Eq. (70), must be modified by subtracting out the pole in addition to the branch cut. We have

$$B^P(J,s)=B(J,s)-\frac{1}{2\pi i}\int_{(m_\rho+m_\pi)^2}^{s_1}\frac{ds'}{s'-s}\times[B(J,s')]_s-\frac{R(J)}{s-s_R(J)}, \tag{74}$$

where $R(J)$ and $s_R(J)$ are the residue and position of the pole. The function $s_R(J)$ is the inverse of the function $\alpha_{\pi\rho}(s)$, that is,

$$s_R(\alpha(s))=s. \tag{75}$$

It follows that $s_R(J)$ and, hence, $B^P(J,s)$ has a branch point at $J=\alpha(9m_\pi^2)$.

Of course, when the partial wave amplitude is constructed from the ratio N/D the discontinuity across this cut vanishes. Because the presence of the branch point depends only on having to subtract the pole term in Eq. (74) and because the partial wave projection of the approximation (35) contains the poles correctly we can see that even the approximate $B^P(J,s)$ contains this branch point. Hence if we can give a good account of the 3π discontinuity of the partial wave amplitude, we can hope to treat this phenomenon in a satisfactory way within the strip approximation. The approximate N and D functions will not, in general, yield a ratio free of this branch point. Nevertheless if our approximations *are* satisfactory it should have a weak effect.

It is worth pointing out that this phenomenon is quite general and that whenever N/D equations result in a D function which lacks a given normal threshold, $s=s_T$, the N and D functions will have branch points in the J plane at

$$J=\alpha_i(s_T) \tag{76}$$

for all trajectories $\alpha_i(s)$.

[55] C. E. Jones, thesis, University of California Radiation Laboratory Report No. UCRL-11125, 1963 (unpublished).

XIV. THE 3π DISCONTINUITY

The 3π discontinuity of the approximate partial-wave amplitude and the corresponding $B^P(J,s)$ is provided entirely by the direct-channel Regge-pole terms. If we suppose that only the poles in the right-half J plane are important, then the discontinuity is

$$\Delta_{3\pi}B^P(J,s)=\sum_i\Delta_{3\pi}\frac{\gamma_{\pi\rho i}(s)}{J-\alpha_i(s)}, \tag{77}$$

where the summation is over the prominent poles and the symbol $\Delta_{3\pi}$ represents the operation of taking the discontinuity.

We have argued, in Sec. XII, that this discontinuity is really important only when a trajectory is passing through the 3π threshold. In a later paper we shall examine in detail $\alpha(s)$ and $\gamma_{\pi\rho}(s)$ near this threshold. We shall find that both functions are essentially real and analytic through the threshold and that the correction to $\alpha(s)$ is proportional to $\gamma_{\pi\rho}(s)\ R(\alpha(s),s)$. The function $R(J,s)$ is determined from a dynamical calculation which utilizes two-body unitarity in the π-π channel and which treats the three π mesons symmetrically. The correction to $\gamma_{\pi\rho}(s)$ is, to the same approximation, still zero. Under these conditions we find that

$$\Delta_{3\pi}\frac{\gamma_{\pi\rho}(s)}{J-\alpha(s)}=\frac{\gamma_{\pi\rho}(s)\Delta_{3\pi}\alpha(s)}{[J-\alpha(s_+)][J-\alpha(s_-)]}, \tag{78}$$

where

$$\alpha(s_\pm)=\alpha(s)\pm\Delta_{3\pi}\alpha(s). \tag{79}$$

Consistent with our approximations we have

$$\Delta_{3\pi}\frac{\gamma_{\pi\rho}(s)}{J-\alpha(s)}=\frac{\gamma_{\pi\rho}(s)\Delta_{3\pi}\alpha(s)}{[J-\alpha(s)]^2}. \tag{80}$$

As indicated above

$$\Delta_{3\pi}\alpha(s)=2i\gamma_{\pi\rho}(s)R(\alpha(s),s) \tag{81}$$

with the result

$$\Delta_{3\pi}B^P(J,s)=2i\sum_i\left(\frac{\gamma_{\pi\rho i}(s)}{J-\alpha_i(s)}\right)^2R(\alpha_i(s),s). \tag{82}$$

The contributions of the direct-channel Regge poles to $B^P(J,s)$ are, then, a left-hand-cut term together with the term

$$\sum_i\frac{1}{\pi}\int_{9m_\pi^2}^{\infty}\frac{ds'}{s'-s}\left(\frac{\gamma_{\pi\rho i}(s')}{J-\alpha_i(s)'}\right)^2R(\alpha_i(s'),s'), \tag{83}$$

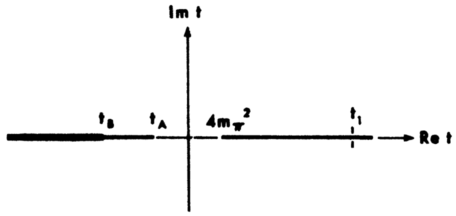

FIG. 19. The partial-wave t-plane singularities when $m_\rho^2<2m_\pi^2$.

where the integration is over the 3π cut as drawn in Fig. 7(b). We emphasize this point since only when so evaluated does $R(\alpha(s),s)$ represent solely residual three-particle effects. This function, in fact, contains a complex π-ρ threshold and when evaluated along the real axis acquires a term proportional to $\rho(\alpha(s),s)$ which swamps any residual three-particle effects. We can see by inspection that the term (83) contains the required branch points at $J=\alpha_i(9m_\pi^2)$.

The expression (83) depends only on $\alpha(s)$ and $\gamma_{\pi\rho}(s)$ together with certain π-π scattering parameters implicit in $R(J,s)$. It follows that if we take for granted the analysis which leads to Eq. (81), the residual 3π effects can be determined entirely in terms of *two*-body parameters and functions.

XV. DYNAMICAL EQUATIONS FOR $b(l,t)$

In order to be able to formulate a complete bootstrap scheme we need equations which will allow us to calculate the residues $\gamma_{\rho\rho}(t)$. These equations are provided by the discontinuity formula for $b(l,t)$.

The analytic structure of $b(l,t)$ is obscured by the overlap of the discontinuities produced by crossed channel singularities and by the anomalous threshold in the direct channel. For example, the cut attached to the branch point $t=t_A$ in Fig. 18 arises both from the anomalous threshold and the π-exchange poles $s=m_\pi^2$, $\bar{s}=m_\pi^2$. In the exact amplitude these overlapping discontinuities are related in a way which allows us to obtain a dynamical equation for $b(l,t)$. Since we do not claim to be able to derive our discontinuity formulas rigorously, we feel free to exhibit this relationship by resorting once again to a continuation in the mass m_ρ^2 along the standard contour, K, in Fig. 15.

We start, then, with a situation in which the ρ meson is stable. The corresponding singularity structure of $b(l,t)$ is shown in Fig. 19. The discontinuity across the

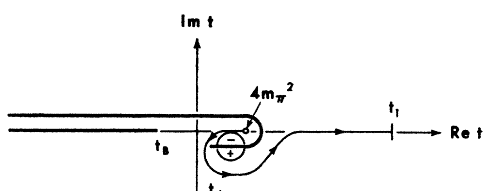

FIG. 20. The partial-wave t-plane singularities when $m_\rho^2>2m_\pi^2$.

π-π normal threshold cut is

$$b(l,t_+)-b(l,t_-)=2i\rho_{\pi\pi}(l,t)b(l,t_+)B_{\pi\pi}(l,t_-). \quad (84)$$

We assume the formula to be valid in the whole strip region $4m_\pi^2\leqq t\leqq t_1$. It follows that

$$b(l,t)=n(l,t)/D_{\pi\pi}(l,t),$$

where $n(l,t)$ has no normal threshold cut. We find that

$$(nl,t)=b^P(l,t)+\frac{1}{\pi}\int_{4m_\pi^2}^{t_1}dt'\frac{b^P(l,t')-b^P(l,t)}{t'-t}\times\rho_{\pi\pi}(l,t')N_{\pi\pi}(l,t'), \quad (85)$$

where

$$b^P(l,t)=b(l,t)-\frac{1}{2\pi i}\int_{4m_\pi^2}^{t_1}\frac{dt'}{t'-t}[b(l,t')]_t \quad (86)$$

$[b(l,t)]_t$ being the normal threshold discontinuity of $b(l,t)$.

As m_ρ^2 is continued past the value $2m_\pi^2$, the left-hand branch point $t=t_A$ moves up round the threshold $t=4m_\pi^2$. Equation (85) can only be maintained if we deform the integration contour to avoid the moving singularity. This situation is illustrated in Fig. 20.

It is more convenient to give up the deformed contour and modify Eq. (85) by the introduction of an anomalous threshold term. We have

$$n(l,t)=b^P(l,t)+\frac{1}{\pi}\int_{4m_\pi^2}^{t_1}dt'\frac{b^P(l,t')-b^P(l,t)}{t'-t}\times\rho_{\pi\pi}(l,t')N_{\pi\pi}(l,t')+\frac{1}{\pi}\int_{t_A}^{4m_\pi^2}\frac{dt'}{t'-t}\times[b^P(l,t')]_t\bar{\rho}_{\pi\pi}(l,t')N_{\pi\pi}(l,t'), \quad (87)$$

where $\bar{\rho}_{\pi\pi}(l,t)$ is $\rho_{\pi\pi}(l,t)$ continued clockwise around the normal threshold and $[b^P(l,t)]_t$ is a suitably evaluated discontinuity of $b^P(l,t)$. Since the singularity $t=t_A$ is produced purely by the π-exchange poles we find

$$[b^P(l,t)]_t=[b^{(\pi)}(l,t)]_t=b^{(\pi)}(l,t_+)-b^{(\pi)}(l,t_-), \quad (88)$$

where $b^{(\pi)}(l,t)$ is the partial-wave projection of the crossed-channel pole terms,

$$b^{(\pi)}(l,t)=\frac{g^2}{4\pi}\frac{1}{2(p_tq_t)^{l+1}}Q_l\left(\frac{\frac{1}{2}t-m_\rho^2}{2p_tq_t}\right). \quad (89)$$

and the limits $\pm$ are shown in Fig. 20.

When m_ρ^2 passes the value $10m_\pi^2$ the singularity t_B also moves around the normal threshold creating a second anomalous contribution to the right side of Eq. (87). We shall not consider this threshold explicitly and so are free to continue m_ρ^2 to its physical value without introducing any further modification to Eq. (87) except the deformation of the anomalous integration path so that it lies along the familiar anomalous threshold cut.

In our scheme, $b^P(l,t)$ is calculated from the partial-wave projection of the approximation (35) by subtracting out the normal and anomalous threshold contributions of the direct-channel pole terms. Since the approximation contains the crossed-channel pole terms, it still predicts the correct anomalous threshold structure for the partial-wave equation.

The residue functions are determined by the equation

$$\gamma_{\rho\rho}(t)=[n(l,t)/(\partial/\partial l)D_{\pi\pi}(l,t)]_{l=\alpha(t)}. \qquad (90)$$

Because of the anomalous contributions to $n(l,t)$ we see that (87) predicts that $\gamma_{\rho\rho}(t)$ will have anomalous thresholds at $t=t_A$ and $t=t_B$ as already stated in Sec. VIII.

We can calculate the discontinuity across the cut attached to $t=t_A$. We find

$$\Delta_A\gamma_{\rho\rho}(t)=2i[b^{(\pi)}(\alpha_{\pi\pi}(t),t)]_t\bar{\rho}_{\pi\pi}(\alpha(t),t)\gamma_{\pi\pi}(t). \qquad (91)$$

Hence, this discontinuity is determined entirely in terms of known functions and the residue functions for π-π scattering. In general, evaluation of the discontinuity of $b^{(\pi)}(l,t)$, though possible, is unilluminating. In order to gain some idea of the behavior of the anomalous discontinuity, $\Delta_A\gamma_{\rho\rho}(t)$, therefore, we assume that $\alpha(t)$ is flat and equal to zero. We have then,

$$[b^{(\pi)}(0,t)]_t =-2\pi i(g^2/4\pi)/[(4m_\pi^2-t)(4m_\rho^2-t)]^{1/2}, \qquad (92)$$

$$\bar{\rho}_{\pi\pi}(0,t)=-i(4m_\pi^2-t)^{1/2}/2(t)^{1/2}, \qquad (93)$$

so that

$$\Delta_A\gamma_{\rho\rho}(t)=-4\pi i[(g^2/4\pi)/[t(4m_\rho^2-t)]^{1/2}]\gamma_{\pi\pi}(t). \qquad (94)$$

Since $\gamma_{\pi\pi}(t)$ is, presumably, appreciable only in the "nearby" strip region, $|t|<t_1$, the same is true of $\Delta_A\gamma_{\rho\rho}(t)$.

In an Appendix we compute the corresponding anomalous threshold discontinuity predicted by the approximation (35) and verify that, asymptotically, it is correct.

XVI. BOOTSTRAP CALCULATION

Clearly we now have all the elements necessary for a bootstrap calculation. If we assume input functions $\alpha^{\text{in}}(\bar{s})$, $\gamma_{\pi\rho}{}^{\text{in}}(\bar{s})$, $\gamma_{\rho\rho}{}^{\text{in}}(t)$ we can calculate, via the approximation (35), the driving term $B^P(J,s)$ for π-ρ scattering and hence obtain, from the dynamical equations of Sec. XII, output functions $\alpha^{\text{out}}(s)$, $\gamma_{\pi\rho}{}^{\text{out}}(s)$. Similarly by assuming input functions $\alpha^{\text{in}}(s)$, $\gamma_{\pi\rho}{}^{\text{in}}(s)$ we can calculate the driving term, $b^P(l,t)$, for the process $\pi+\pi\rightarrow\rho+\rho$ and obtain output functions $\gamma_{\rho\rho}{}^{\text{out}}(t)$. A bootstrap solution consists in finding a set of functions which when used as input reproduce themselves as output.

We are faced with the practical problem of parametrizing the input functions. If we continue to use the Froissart-Gribov formulas, in Eqs. (53) and (59), to effect the continuation to complex J, then, we require parametrizations accurate in regions where the input functions have singularities. By converting, partially, to the Wong method[9] for continuing in angular momentum, we shall avoid the necessity of constructing such parametrizations.

First we rewrite Eq. (52), in an obvious notation, as

$$A_{\pi\rho}(s,t,\bar{s})=A_{\pi\rho}{}^N(s,t)+\frac{1}{\pi}\int_C\frac{dt'}{t'-t}D_t{}^A(t',s)+A_{\pi\rho}{}^N(s,\bar{s}) -\frac{g^2}{s-m_\pi^2}-\frac{g^2}{\bar{s}-m_\pi^2}. \qquad (95)$$

The superfix N on the first and third terms on the right side emphasizes that they contain no anomalous thresholds. We have then, instead of Eq. (53),

$$\begin{aligned}(2q_s^2)A^{(\pm)}(J,s)=&\frac{1}{8\pi}\int_{-\infty}^{0}dt'[Q_J(z')]_zA_{\pi\rho}{}^N(s,t')\\&+\frac{1}{8\pi}\int_C dt'Q_J(z')D_t{}^A(t',s)\\&\pm\frac{1}{8\pi}\int_{-\infty}^{\bar{s}_0}d\bar{s}'[Q_J(z'')]_zA_{\pi\rho}{}^N(s,\bar{s}')\\&\pm\frac{g^2}{8\pi}Q_J\left(\frac{2p_0q_0-m_\rho^2}{2q_s^2}\right),\end{aligned} \qquad (96)$$

where

$$\bar{s}_0=2(m_\rho^2+m_\pi^2)+\lambda(s,m_\rho^2,m_\pi^2)/s+s, \qquad (97)$$

and $[Q_J(z)]_z$ represents the z discontinuity of $Q_J(z)$ divided by $2i$.

Notice that the $\bar{s}'$ integration is now over a region, which, for the most part, lies below the singularities of $\alpha(\bar{s})$ and $\gamma_{\pi\rho}(\bar{s})$ although, for $s<(m_\rho^2-m_\pi^2)^2/9m_\pi^2$, the end point, $\bar{s}_0$, does lie above the 3π threshold. However, since we regard the 3π discontinuity as a weak effect, we can try to parameterize the input functions by formulas which lack it. Thus we could have, for example,

$$\alpha^{\text{in}}(\bar{s})=a+\frac{b_2}{\bar{s}-\bar{s}_2}+\frac{b_3}{\bar{s}-\bar{s}_3},$$

$$\gamma_{\pi\rho}{}^{\text{in}}(\bar{s})=\frac{c_2}{\bar{s}-\bar{s}_2}+\frac{c_3}{\bar{s}-\bar{s}_3}, \qquad (98)$$

where we expect $\bar{s}_2$ and $\bar{s}_3$ to lie towards the upper end of the strip. If we use such formulas $B(J,s)$ will lose the geometrically most prominent part of the analytic structure of Fig. 16(a), which means that the driving term $B^P(J,s)$ will be correspondingly simpler to evaluate.

To achieve a parametrization of $\gamma_{\rho\rho}{}^{\text{in}}$ we first split the residue functions into two parts. Thus,

$$\gamma_{\rho\rho}(t)=\gamma_{\rho\rho}{}^N(t)+\gamma^A{}_{\rho\rho}(t), \qquad (99)$$

where $\gamma_{\rho\rho}{}^N(t)$ has discontinuities only above the π-π threshold and $\gamma_{\rho\rho}{}^A(t)$ contains only the anomalous threshold discontinuities. The important point is that $A_{\pi\rho}{}^N(s,t)$ depends only on $\gamma_{\rho\rho}{}^N(t)$ while $D_t{}^A(t',s)$ is determined by the discontinuity of $\gamma_{\rho\rho}{}^A(t)$. Since the t' integration in Eq. (96) lies entirely in a region where $\gamma_{\rho\rho}{}^N(t)$ has no singularities, we can parametrize $\gamma_{\rho\rho}{}^N(t)$ as

$$\gamma_{\rho\rho}{}^N(t)=\frac{d_2}{t-t_2}+\frac{d_3}{t-t_3}. \tag{100}$$

Again, replacing Eq. (59) by a Wong formula would free us, by arguments of the same type as above, to use the forms suggested in Eq. (98) for $\alpha_{\pi\rho}{}^{\text{in}}$ and $\gamma_{\pi\rho}{}^{\text{in}}$ in calculating the driving term $b^P(l,t)$. In this case, also, we achieve a simplification of the analytic structure; by removing the 3π discontinuity we eliminate the anomalous threshold $t=t_B$, thus simplifying, in turn, the calculation of the discontinuity of $\gamma_{\rho\rho}{}^A(t)$.

As is usual, in a bootstrap calculation, we are not guaranteed in advance that the output functions we obtain will be suitable for use as input functions. In our case this difficulty takes the acute form that our output trajectories may not be real analytic. Similar remarks apply to the residue functions.

Another consistency requirement which is not guaranteed is that $\gamma_{\pi\rho}{}^{(\pi)}(m_\pi{}^2)=g^2$. This is an example of the familiar fact that a bootstrap theory overdetermines the scattering parameters. We can only hope that, in a physical situation, our approximation scheme does not depart too seriously from reality and that these consistency requirements are not too badly violated in practice.

XVII. CONCLUSION

We have discussed in this paper how a bootstrap calculation might be performed when one of the external particles in the scattering amplitude is unstable. The important properties of the unstable amplitude of which we were able to give a discussion were

(i) the complex normal threshold discontinuity,
(ii) residual three-particle effects,
(iii) the prominent anomalous thresholds which arise from the instability of the external particle, and
(iv) asymptotic behavior.

Many problems remain, however, and because of this our discussion must be regarded as provisional. Of these problems perhaps the most important is that of removing the restriction to S-wave resonances. Of course this problem is not unique to unstable amplitudes. External particles of high spin,[29] stable or unstable, cause difficulties in the formulation of dynamical scattering equations. Mandelstam has pointed out that the resolution of this difficulty lies in a better understanding of multiparticle effects.[30]

Further difficulties are, the construction of satisfactory Regge-pole terms for particles of unequal masses and the estimation of the importance of analytic structure entirely omitted in the approximation scheme. There seems to be no way of performing this estimation except in the context of a numerical calculation.

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APPENDIX

In this Appendix we utilize the anomalous discontinuity, obtained in Sec. XV for the function $\gamma_{\rho\rho}(t)$, to evaluate the anomalous discontinuity predicted for $A_{\pi\rho}(s,t,\bar{s})$ by the approximation (35). This discontinuity receives contributions only from the t-reaction Regge-pole terms. In order to simplify the writing we assume that only one Regge-pole term is important. We find

$$\Delta_A A_{\pi\rho}(s,t,\bar{s})=8\pi\Delta_A[R_{\rho\rho}{}^{s_1}(t,s)+R_{\rho\rho}{}^{s_1}(t,\bar{s})]. \tag{A1}$$

From Eq. (39) of Sec. VII, we see that

$$\Delta_A R_{\rho\rho}{}^{s_1}(t,s)=\frac{1}{\pi}\int_{s_1}^{\infty}\frac{ds'}{s'-s}\Delta_A R_{\rho\rho}(s',t), \tag{A2}$$

where

$$\Delta_A R_{\rho\rho}(s',t) =\tfrac{1}{2}\pi[2\alpha_{\pi\pi}(t)+1](-p_tq_t)[\Delta_A\gamma_{\rho\rho}(t)]P_{\alpha(t)}(-x') \tag{A3}$$

using Eq. (91) of Sec. XV we find

$$\Delta_A R_{\rho\rho}(s',t)=2i[b^{(\pi)}(\alpha(t),t)]_t\bar{\rho}_{\pi\pi}(\alpha(t),t)\tilde{R}_{\pi\pi}(s',t), \tag{A4}$$

where

$$\tilde{R}_{\pi\pi}(s',t)=\tfrac{1}{2}\pi[2\alpha(t)+1] \times(-p_tq_t)^{\alpha(t)}\gamma_{\pi\pi}(t)P_{\alpha(t)}(-x') \tag{A5}$$

so that

$$\Delta_A R_{\rho\rho}{}^{s_1}(t,s)=2i[b^{(\pi)}(\alpha(t),t)]_t \times\bar{\rho}_{\pi\pi}(\alpha(t),t)\tilde{R}_{\pi\pi}{}^{s_1}(t,s), \tag{A6}$$

where

$$\tilde{R}_{\pi\pi}{}^{s_1}(t,s)=\frac{1}{\pi}\int_{s_1}^{\infty}\frac{ds'}{s'-s}\tilde{R}_{\pi\pi}(s',t). \tag{A7}$$

Hence $\tilde{R}_{\pi\pi}(t,s)$ is a modified ρ-ρ Regge-pole term with residues $\frac{1}{2}(p_tq_t)^{\alpha(t)}\gamma_{\pi\pi}(t)$. We find, then, that

$$(1/8\pi)\Delta_A A_{\pi\rho}(s,t,\bar{s})=2i[b^{(\pi)}(\alpha(t),t)]_t\bar{\rho}_{\pi\pi}(\alpha(t),t) \times[\tilde{R}_{\pi\pi}{}^{s_1}(t,s)+\tilde{R}_{\pi\pi}{}^{s_1}(t,\bar{s})]. \tag{A8}$$

We can verify that the right side of (A8) yields asymptotically the correct anomalous discontinuity in the following way.

First we note that the π-π normal threshold discontinuity contains the complete anomalous threshold term. We have for this discontinuity

$$A_{\pi\rho}(s,t_+,\bar{s})-A_{\pi\rho}(s,t_-,\bar{s})$$
$$=i(2\pi)^4\int d\rho(2)\ A_{\pi\pi}(t_-,s')A_{\pi\rho}(s'',t_+,\bar{s}'')\,, \quad \text{(A9)}$$

where

$$d\rho(2)=[1/(2\pi)^6][q_t/4(t)^{1/2}]d\Omega'. \quad \text{(A10)}$$

The contribution $T(t,s)$ which contains the anomalous threshold is obtained by replacing $A_{\pi\rho}(s,t,\bar{s})$ by the π-exchange poles. Because the anomalous threshold lies on the second sheet of $A_{\pi\rho}$ when $m_\rho^2<2m_\pi^2$, an additional minus sign is introduced. We have, then, for $T(t,s)$,

$$T(t,s)=-i(2\pi)^4\int d\rho(2)\ A_{\pi\pi}(t_-,s')g^2 \times\left[\frac{1}{m_\pi^2-s''}+\frac{1}{m_\pi^2-\bar{s}''}\right]. \quad \text{(A11)}$$

That part of the right-hand side which contains only right-hand s-plane cuts is

$$T^{(+)}(t,s)=-i(2\pi)^4\int d\rho(2) \times A_{\pi\rho}{}^{(+)}(t_-,s')g^2\frac{1}{m_\pi^2-s''}. \quad \text{(A12)}$$

It we expand both factors of the integrand in a Legendre series, we obtain

$$A_{\pi\pi}{}^{(+)}(t_+,s')=8\pi\sum_{l=0}^{\infty}(2l+1)(q_t^2)^l B_{\pi\pi}(l,t)P_l(x')\,,$$

$$\frac{g^2}{m_\pi^2-s''}=4\pi\sum_{l=0}^{\infty}(2l+1)(p_tq_t)^l b^{(\pi)}(l,t)P_l(x'')\,. \quad \text{(A13)}$$

It follows that

$$T^{(+)}(t,s)=-i8\pi\sum_{l=0}^{\infty}(2l+1)\rho_{\pi\pi}(l,t)B_{\pi\pi}(l,t_-)(p_tq_t)^l \times b^{(\pi)}(l,t)P_l(x)\,. \quad \text{(A14)}$$

If we perform a Sommerfeld-Watson transformation on this series we find, ignoring the background integral, that

$$T^{(+)}(t,s)=i(8\pi)B_{\pi\pi}(\alpha(t),t)\gamma_{\pi\pi}(t)(p_tq_t)^{\alpha(t)} \times b^{(\pi)}(\alpha(t),t)P_{\alpha(t)}(-x)\,. \quad \text{(A15)}$$

That is,

$$T^{(+)}(t,s)=2i(8\pi)b^{(\pi)}(\alpha(t),t)\rho_{\pi\pi}(\alpha(t),t)\mathcal{R}_{\pi\pi}(t,s)\,, \quad \text{(A16)}$$

where

$$\mathcal{R}_{\pi\pi}(t,s)=\frac{\pi}{2}\frac{(2\alpha(t)+1)}{\sin\pi\alpha(t)}(p_tq_t)^{\alpha(t)}\gamma_{\pi\pi}(t)P_{\alpha(t)}(-x)\,. \quad \text{(A17)}$$

The only factor on the right side of Eq. (A16) with the anomalous threshold is $b^{(\pi)}(\alpha(t),t)$. When we take the discontinuity we find

$$\Delta_A T^{(+)}(t,s)=2i(8\pi)[b^{(\pi)}(\alpha(t),t)]_t \times\bar{\rho}_{\pi\pi}(\alpha(t),t)\mathcal{R}_{\pi\pi}(t,s)\,. \quad \text{(A18)}$$

It follows then that the discontinuity in the full amplitude is given by

$$\frac{1}{8\pi}\Delta_A A_{\pi\rho}(s,t,\bar{s})=2i[b^{(\pi)}(\alpha(t),t)]_t\bar{\rho}_{\pi\pi}(\alpha(t),t) \times[\mathcal{R}_{\pi\pi}(t,s)+\mathcal{R}_{\pi\pi}(t,\bar{s})] \quad \text{(A19)}$$

since asymptotically

$$\tilde{R}_{\pi\pi}{}^{a1}(t,s)\sim\mathcal{R}_{\pi\pi}(t,s)$$

we see that Eqs. (A8) and (A19) yield the same result when s is large.